

PARTIAL FLAG INCIDENCE ALGEBRAS

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ABSTRACT. The n^{th} partial flag incidence algebra of a poset P is the set of functions from P^n to some ring which are zero on non-partial flag vectors. These partial flag incidence algebras for $n > 2$ are not commutative, not unitary, and not associative. However, partial flag incidence algebras contain generalized zeta, delta, and Möbius functions which contain delicate information of the poset. Using these functions we define multi-indexed Whitney numbers, of both kinds, and develop relationships between them. Then we apply these results to recursively construct a closed formula for the coefficients of the Kazhdan-Lusztig polynomial for a matroid. We also study some generalized characteristic polynomials of posets which are not evaluations of Tutte polynomials.

1. INTRODUCTION

Let \mathcal{P} be a locally finite poset. The incidence algebra of \mathcal{P} with ring R , here written as $\mathcal{I}^2(\mathcal{P}, R)$, is the set of functions from \mathcal{P}^2 to R which are zero on elements (X, Y) where $X \not\leq Y$. Addition in $\mathcal{I}^2(\mathcal{P}, R)$ is defined as $(f + g)(X, Y) = f(X, Y) + g(X, Y)$ and the product is given by convolution

$$(f * g)(X, Y) = \sum_{X \leq Z \leq Y} f(X, Z)g(Z, Y)$$

where the juxtaposition above is the product in R . These classical incidence algebras contain some of the most famous combinatorial and number theoretic invariants like the Möbius function, the chromatic polynomial of graphs, and the Tutte polynomial. The book [21] provides an excellent reference for incidence algebras. The main character we study here is a suggestion for a generalization of these incidence algebras, written here as $\mathcal{I}^n(\mathcal{P}, R)$ and defined in Section 2.

The classical incidence algebras provide excellent examples of associative, unital, but non-commutative algebras. These generalized incidence algebras $\mathcal{I}^n(\mathcal{P}, R)$ are none of these. We call them partial flag incidence algebras because as a set they are functions from the set of partial n flags to a ring R and the product is a generalization of the convolution. These algebras contain many interesting invariants including generalizations of the classical Möbius functions and characteristic polynomials.

Using generalized zeta and Möbius functions we define generalized Whitney numbers of both kinds. The Whitney numbers of the second kind are key players in the study of the

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cd-index and are usually grouped together into one function called the flag f-vector (for example see [22]). There are many open conjectures about Whitney numbers (see [20], [14], and [5]). There are also some astonishing results, like the generalized “Hyperplane Theorem” (see [5]), and some recent, like the log-concavity conjecture (see [11]). In Section 4 we collect a few lemmas on computing sums of Whitney numbers and demonstrate a formula for writing Whitney numbers of the first kind in terms of a sum of Whitney numbers of the second kind. As a first application in Corollary 4.5 we apply this formula to Zaslavsky’s face count formulas in [24]. The main application is to the Kazhdan-Lusztig polynomial of a matroid.

Motivation for this study comes from trying to develop a formula for the coefficients for the “matroid Kazhdan-Lusztig polynomials” defined in [7]. If the matroid is realizable over \mathbb{C} then the matroid Kazhdan-Lusztig polynomial is the intersection cohomology Poincaré polynomial of the associated reciprocal plane (again see [7]). These polynomials are notoriously difficult to compute in general. For example, the main focus in [17] is the intersection cohomology for uniform matroids of rank n on $n + 1$ elements. In [8] an equivariant matroid Kazhdan-Lusztig polynomial is defined and impressively used to find a formula for the ordinary matroid Kazhdan-Lusztig polynomial for all uniform matroids. The main result, Theorem 5.4, in this paper is a formula in terms of “multi-indexed Whitney numbers” of the second kind (see Section 3) for the coefficients of the matroid Kazhdan-Lusztig polynomial. In [7] the degree 1 and 2 coefficients of the Kazhdan-Lusztig polynomial of a matroid are written in terms of doubly-indexed Whitney numbers of the second kind (see [9]). In Section 5 we generalize this formula to all coefficients by utilizing the results in Section 3 on the generalized Whitney numbers.

There have been quite a few generalizations of incidence algebras in the past, see [15, 23, 19, 10]. Though the algebras considered here seem to be new. Koszulness of incidence algebras was studied in [18]. It would be interesting to see if any of this makes sense in this more general setting. Also the classical incidence algebras have a nice Hopf algebra structure (see [6]). At this time the author has not consider this direction. Another important result for incidence algebras is the classification of reduced incidence algebras in [13]. It would be interesting to see if any of that theory generalizes to $\mathcal{I}^n(\mathcal{P}, R)$.

Last we study a partial flag version of the characteristic polynomial. If the poset is the graph partitions poset of a graph then this gives a generalized chromatic polynomial. So, far this polynomial seems to be very mysterious. First we compute these polynomials for the Boolean lattice. Then we use this to demonstrate that in general these polynomials will not satisfy any kind of deletion-contraction formula. Hence these polynomials are new invariants and not evaluations of Tutte polynomials. Various generalizations of characteristic polynomials have been considered in the past. For example those studied in [3], but these are evaluations of Tutte polynomials.

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2. FLAG INCIDENCE ALGEBRAS

For incidence algebras we need to restrict to the class of locally finite posets \mathcal{P} . This means that any interval $[a, b] = \{x \in \mathcal{P} | a \leq x \leq b\}$ is a finite set. This is not actually a big restriction since most of the posets we consider are finite.

Definition 2.1. Let \mathcal{P} be a locally finite poset. The n^{th} *flag incidence algebra* on \mathcal{P} , for $n \geq 2$, with coefficients in a ring R is $\mathcal{I}^n(\mathcal{P}, R)$ (we will suppress the R when it is clear from context) the set of all functions $f : \mathcal{F}^n(\mathcal{P}) \rightarrow R$ where

$$\mathcal{F}^n(\mathcal{P}) = \{(X_1, \dots, X_n) \in \mathcal{P}^n | X_1 \leq X_2 \leq \dots \leq X_n\}.$$

Addition in $\mathcal{I}^n(\mathcal{P})$ is given by

$$(f + g)(X_1, \dots, X_n) = f(X_1, \dots, X_n) + g(X_1, \dots, X_n)$$

and multiplication is given by a convolution

$$(f * g)(X_1, \dots, X_n) = \sum_{X_i \leq Y_i \leq X_{i+1}} f(X_1, Y_1, Y_2, \dots, Y_{n-1})g(Y_1, Y_2, \dots, Y_{n-1}, X_n)$$

where the juxtaposition above is just multiplication in the ring R .

The choice of this multiplication is particularly suited to our needs as we will see below. Unfortunately for $n > 2$ our choice makes $\mathcal{I}^n(\mathcal{P})$ a little messy when compared to $n = 2$.

Now we define some particularly important elements of $\mathcal{I}^n(\mathcal{P})$.

Definition 2.2. (1) For an ordered subset $I = \{i_1, \dots, i_s\} \subseteq [n]$ the *piece-wise delta function* is $\delta_I \in \mathcal{I}^n(\mathcal{P})$ defined by

$$\delta_I(X_1, \dots, X_n) = \begin{cases} 1 & X_{i_1} = X_{i_2} = \dots = X_{i_s} \\ 0 & \text{else} \end{cases}.$$

(We will usually suppress the set distinguishing brackets on I .)

(2) For subset of flags $S \subseteq \mathcal{F}^n(\mathcal{P})$ the *characteristic function* $C_S \in \mathcal{I}^n(\mathcal{P})$ with respect to S is defined by

$$C_S(X_1, \dots, X_n) = \begin{cases} 1 & (X_1, \dots, X_n) \in S \\ 0 & \text{else} \end{cases}.$$

(3) The k^{th} -zeta function ζ_k on \mathcal{P} is the constant function 1 on $\mathcal{F}^n(\mathcal{P})$, so for all $(X_1, \dots, X_k) \in \mathcal{F}^n(\mathcal{P})$

$$\zeta_k(X_1, \dots, X_k) = 1.$$

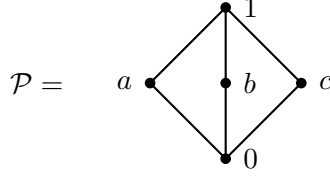


FIGURE 1. A poset where the left and right Möbius functions are different.

- (4) The k^{th} -Möbius function on \mathcal{P} is $\mu_k : \mathcal{F}^k(\mathcal{P}) \rightarrow \mathbb{Z}$ recursively defined by $\mu_k(X_1, \dots, X_k) = 1$ if $X_1 = \dots = X_k$ and

$$\sum \mu_k(X_1, Y_1, \dots, Y_{k-1}) = 0$$

where the sum is over all k -tuples where $X_1 \leq Y_1 \leq X_2 \leq Y_2 \leq X_3 \leq \dots \leq Y_{k-1} \leq X_k$.

Remark 2.3. Note that we can think of $\zeta_k = \delta_1$. And if $n = 2$ then the usual delta function is $\delta = \delta_{[2]}$ in our notation.

This definition of the Möbius function is very particular. Note that it is defined as the element a such that $a * \zeta = \delta_{[k]}$ which is a generalization of the usual Möbius function when $k = 2$. Since $\delta_{[k]}$ is not a unit and $\mathcal{I}^k(\mathcal{P}, R)$ is not commutative there is another element μ^r defined by $\zeta * \mu^r = \delta_{[k]}$. In the next example we show that these two functions are different.

Example 2.4. Let \mathcal{P} be the poset in Figure 1. Then $\mu(0, a, 1) = -2$ and hence $\mu(0, 1, 1) = 4$. But $\mu^r(0, 1, 1) = 2$. This example comes up also as the counter-example to a generalized deletion-restriction formula for the higher characteristic polynomials.

The fact that μ and μ^r are different seems to indicate the non-triviality of the invariants studied here. We choose to study the first μ exclusively since it suits the generalization of characteristic polynomials presented in Section 6.

Now we turn to fundamental properties of the incidence algebra $\mathcal{I}^k(\mathcal{P}, R)$. The proof of the next proposition maybe much easier than the one supplied here, none the less it is elementary.

Proposition 2.5. *The incidence algebra $\mathcal{I}^n(\mathcal{P})$ is not associative for $n > 2$.*

Proof. To do this we use the functions $\delta_{1,2}$, $\delta_{2,3}$, and $\zeta := \zeta_n$. First we compute

$$\begin{aligned}
 (1) \quad ((\delta_{1,2} * \delta_{2,3}) * \zeta)(X_1, \dots, X_n) &= \sum_{X_i \leq Y_i \leq X_{i+1}} \delta_{1,2} * \delta_{2,3}(X_1, Y_1, \dots, Y_{n-1}) \\
 &= \sum_{X_i \leq Y_i \leq X_{i+1}} \left[\sum_{Y_{i-1} \leq Z_i \leq Y_i} \delta_{1,2}(X_1, Z_1, Z_2, \dots, Z_{n-1}) \delta_{2,3}(Z_1, Z_2, Z_3, \dots, Z_{n-1}, Y_{n-1}) \right] \\
 &= \sum_{X_i \leq Y_i \leq X_{i+1}} |[Y_3, Y_4, \dots, Y_{n-1}]|
 \end{aligned}$$

where $[Y_3, \dots, Y_{n-1}] = \{[s_1, \dots, s_{n-3}] \in \mathcal{F}^{n-3} | Y_{i+2} \leq s_i \leq Y_{i+3}\}$. Now we compute

$$\begin{aligned}
 (2) \quad (\delta_{1,2} * (\delta_{2,3} * \zeta))(X_1, \dots, X_n) &= \sum_{X_i \leq Y_i \leq X_{i+1}} (\delta_{2,3} * \zeta)(X_1, Y_2, \dots, Y_{n-1}, X_n) \\
 &= \sum_{X_i \leq Y_i \leq X_{i+1}} \left[\sum_{\substack{Y_{i-1} \leq Z_i \leq Y_i \\ Y_{n-1} \leq Z_{n-1} \leq X_n}} \delta_{2,3}(X_1, Z_1, \dots, Z_{n-1}) \right] \\
 &= \sum_{X_i \leq Y_i \leq X_{i+1}} |[Y_3, \dots, Y_{n-1}, X_n]|.
 \end{aligned}$$

Note that the element $\delta_{2,3}$ is not even defined unless $n > 2$. As long as $X_{n-1} < X_n$ we have that (1) is strictly less than (2) for $n > 2$ and this finishes the proof. \square

Next we will prove that \mathcal{I}^n is not unital for $n > 2$.

Proposition 2.6. *The incidence algebra $\mathcal{I}^n(\mathcal{P})$ does not have a one sided unit for $n > 2$.*

Proof. Suppose there was a right unit $u \in \mathcal{I}^n(\mathcal{P})$. Compute

$$(3) \quad (\delta_{1,n} * u)(X_1, \dots, X_n) = u(X_1, X_1, X_1, \dots, X_1, X_n) = \begin{cases} 1 & X_1 = X_n \\ 0 & \text{else} \end{cases}$$

and

$$(4) \quad (\delta_{n-2,n-1} * u)(X_1, \dots, X_1, X_n, X_n) = u(X_1, X_1, \dots, X_1, X_n, X_n) = \begin{cases} 1 & X_1 = X_n \\ 0 & \text{else} \end{cases}.$$

Note that the last expression (4) is only defined when $n > 2$. Also, suppose that X_n covers X_1 , so that $X_1 < X_n$ and there does not exist Y such that $X_1 < Y < X_n$. Then both expressions (3) and (4) are zero. But

$$\begin{aligned}
 \zeta * u(X_1, \dots, X_1, X_1, X_n) &= \sum_{X_1 \leq Y \leq X_n} u(X_1, \dots, X_1, Y, X_n) \\
 &= u(X_1, \dots, X_1, X_n) + u(X_1, \dots, X_1, X_n, X_n) = 1
 \end{aligned}$$

which is a contradiction. The left side is similar it just uses the computations $u * \delta_{1,n}(X_1, \dots, X_n)$, $u * \delta_{2,3}(X_1, X_1, X_n, \dots, X_n)$, and $u * \zeta(X_1, X_n, \dots, X_n)$. \square

So far this all seems to be bad news for these higher dimensional incidence algebras. But we do have a very natural product formula generalizing the usual formula (see Proposition 2.1.12 in [21]).

Proposition 2.7. *If P and Q are finite posets then*

$$\mathcal{I}^n(P \times Q, R) \cong \mathcal{I}^n(P, R) \otimes_R \mathcal{I}^n(Q, R).$$

Proof. We define a map $\varphi : \mathcal{I}^n(P, R) \otimes_R \mathcal{I}^n(Q, R) \rightarrow \mathcal{I}^n(P \times Q, R)$ on simple tensors by

$$\varphi(g \otimes h)((X_1, Y_1), \dots, (X_n, Y_n)) = g(X_1, \dots, X_n)h(Y_1, \dots, Y_n)$$

where the product on the right hand side is just multiplication in the ring R . First we show that φ is a ring homomorphism. To shorten the notation let $(\bar{X}, \bar{Y}) := ((X_1, Y_1), \dots, (X_n, Y_n))$. On simple tensors

$$\begin{aligned} \varphi((e \otimes f) \cdot (g \otimes h))(\bar{X}, \bar{Y}) &= \varphi(e *_P g \otimes f *_Q h)(\bar{X}, \bar{Y}) \\ &= (e *_P g)(\bar{X}, \bar{Y})(f *_Q h)(\bar{X}, \bar{Y}) \\ &= \left(\sum_{\bar{X} \leq \bar{W}} e(\bar{W})g(\bar{W}) \right) \left(\sum_{\bar{Y} \leq \bar{Z}} f(\bar{Z})h(\bar{Z}) \right) \\ &= \sum_{\bar{X} \leq \bar{W}} \sum_{\bar{Y} \leq \bar{Z}} e(\bar{W})f(\bar{Z})g(\bar{W})h(\bar{Z}) \\ &= \sum_{\bar{X} \leq \bar{W}} \sum_{\bar{Y} \leq \bar{Z}} \varphi(e \otimes f)(\bar{W}, \bar{Z})\varphi(g \otimes h)(\bar{W}, \bar{Z}) \\ &= (\varphi(e \otimes f) *_P \times_Q \varphi(g \otimes h))(\bar{X}, \bar{Y}) \end{aligned}$$

where in the third, fourth, and fifth lines above the inequality $\bar{X} \leq \bar{W}$ means the usual $X_1 \leq W_1 \leq X_2 \leq W_2 \leq \dots \leq W_{n-1} \leq X_n$ and $e(\bar{W})g(\bar{W})$ means the usual

$$e(X_1, W_1, \dots, W_{n-1})g(W_1, \dots, W_{n-1}, X_n).$$

Then a routine check on non-simple tensors shows φ is a ring homomorphism.

The fact that φ is injective is nearly a tautology. Surjectivity is more interesting and there we use the finiteness hypothesis of P and Q . Let $f \in \mathcal{I}^n(P \times Q, R)$. Then define

$$F = \sum_{\bar{X} \in \mathcal{F}^n(P)} \sum_{\bar{Y} \in \mathcal{F}^n(Q)} f((X_1, Y_1), \dots, (X_n, Y_n)) C_{(X_1, \dots, X_n)} \otimes C_{(Y_1, \dots, Y_n)}$$

which is well defined since the posets are finite and $F \in \mathcal{I}^n(P, R) \otimes_R \mathcal{I}^n(Q, R)$. Since $\varphi(C_{(X_1, \dots, X_n)} \otimes C_{(Y_1, \dots, Y_n)}) = C_{((X_1, Y_1), \dots, (X_n, Y_n))}$ we have that $\varphi(F) = f$. \square

3. WHITNEY NUMBERS

Using the functions μ_k and ζ_k we can define multi-indexed Whitney numbers. A good reference for classical Whitney numbers is [1] and these were generalized to 2 subscripts in [9].

Definition 3.1. Let \mathcal{P} be a ranked locally finite poset. Let $I = \{i_1, \dots, i_k\}$ be an ordered k -tuple such that for all j , $i_j \in \{0, 1, 2, \dots, \text{rk}\mathcal{P}\}$.

- (1) The *multi-indexed Whitney numbers of the first kind* are

$$w_I(\mathcal{P}) = \sum \mu_k(X_1, X_2, \dots, X_k)$$

where the sum is over all k -tuples (X_1, \dots, X_k) where $X_1 \leq X_2 \leq \dots \leq X_k$ and for all $j \in [k]$, $\text{rk}X_j = i_j$.

- (2) The *multi-indexed Whitney numbers of the second kind* are

$$W_I(\mathcal{P}) = \sum \zeta_k(X_1, X_2, \dots, X_k)$$

where the sum is over all k -tuples (X_1, \dots, X_k) where $X_1 \leq X_2 \leq \dots \leq X_k$ and for all $j \in [k]$, $\text{rk}X_j = i_j$.

When the context of the poset is clear we will just write W_I instead of $W_I(\mathcal{P})$.

Remark 3.2. Note that the original Whitney numbers, w_i , of the first kind in our notation is just $w_{0,i}$. The Whitney numbers of the second kind have the funny property that some indices are trivial in the sense that with or without these indices we have the same value. For example, if \mathcal{P} is a lattice then $W_{0,i} = W_i$.

Remark 3.3. Let \mathcal{P} be a rank n poset with $\hat{0}$ and $\hat{1}$. Let $[n] = \{0, 1, \dots, n\}$. We can make a function $\alpha : [n] \rightarrow \mathbb{Z}$ defined by $\alpha(I) = W_I$. The function α is called the *flag f -vector* in the literature (see [12] and [22]). This function is used in the original definition of the so called *cd-index* of the poset \mathcal{P} . It would be interesting to know if there were any connection between some of the facts presented in this study and various results on the cd-index.

3.1. Boolean Lattices. Let \mathcal{B}_n be the rank n Boolean lattice. In the following lemma we compute the k^{th} Möbius function on elements in \mathcal{B}_n . We can prove the following by a quick application of Proposition 2.7. However, we give a different proof which is much longer but instructive for how one might deal with computing the Möbius function in general.

Proposition 3.4. If $X_1 \leq \dots \leq X_k \in \mathcal{B}_n$ then

$$\mu_k(X_1, \dots, X_k) = (-1)^{\text{rk}(X_1) + \dots + \text{rk}(X_k)}.$$

Proof. We induct on n . The base case is $n = 1$. Since as a set we can write $\mathcal{B}_1 = \{0, 1\}$ we have that the only possible Möbius values are

$$b_i := \mu_k(0, \dots, 0, 1, \dots, 1)$$

where there are i 1's. We induct again on i . The case $i = 0$ is by definition. Then the recursion gives that $0 = b_i + b_{i-1} = b_i + (-1)^{i-1}$. Hence $b_i = (-1)^i$. Now we use the product formula since we know that $\mathcal{B}_n \cong \mathcal{B}_{n-1} \times \mathcal{B}_1$. For $(X_1, \dots, X_k) \in \mathcal{F}l^k(\mathcal{B}_n)$

let $((Y_1, Z_1), \dots, (Y_k, Z_k)) \in \mathcal{F}^k(\mathcal{B}_{n-1} \times \mathcal{B}_1)$ be the corresponding element. Then using induction

$$\begin{aligned}\mu_k^{\mathcal{B}_n}(X_1, \dots, X_k) &= \mu_k^{\mathcal{B}_{n-1}}(Y_1, \dots, Y_k) \mu_k^{\mathcal{B}_1}(Z_1, \dots, Z_k) \\ &= (-1)^{\text{rk}(Y_1) + \dots + \text{rk}(Y_k)} (-1)^{\text{rk}(Z_1) + \dots + \text{rk}(Z_k)}.\end{aligned}$$

Since $\text{rk}(Y_i) + \text{rk}(Z_i) = \text{rk}(X_i)$ we have finished the proof. \square

Corollary 3.5. *If $I = (i_1, \dots, i_k)$ then $w_I(\mathcal{B}_n) = (-1)^{i_1 + \dots + i_k} W_I(\mathcal{B}_n)$.*

Next we compute these numbers for the Boolean lattice. For any ranked poset \mathcal{P} we let $\mathcal{P}(j) = \{X \in \mathcal{P} | \text{rk}(X) = j\}$.

Proposition 3.6. *If $I = (i_1, \dots, i_k)$ then*

$$W_I(\mathcal{B}_n) = \binom{n}{i_1, i_2 - i_1, i_3 - i_2, \dots, i_k - i_{k-1}, n - i_k}.$$

Proof. We start at the bottom of the chain $i_1 \leq \dots \leq i_k$. There are exactly $\binom{n}{i_1}$ elements of rank i_1 in \mathcal{B}_n (i.e. $|\mathcal{B}_n(j)| = \binom{n}{j}$). Then for any $X \in \mathcal{B}_n(i_1)$ the restriction to X is $\mathcal{B}_n^X \cong \mathcal{B}_{n-i_1}$ and the elements above X of rank i_2 in \mathcal{B}_n are now of rank $i_2 - i_1$ in \mathcal{B}_n^X . So, for every $X \in \mathcal{B}_n(i_1)$ the number of elements above it is $\binom{n-i_1}{i_2-i_1}$. In general for every $Y \in \mathcal{B}_n(i_j)$ there are $\binom{n-i_j}{i_{j+1}-i_j}$ above it in $\mathcal{B}_n(i_{j+1})$. Hence

$$W_I(\mathcal{B}_n) = \binom{n}{i_1} \binom{n-i_1}{i_2-i_1} \dots \binom{n-i_{k-1}}{i_k-i_{k-1}} = \binom{n}{i_1, i_2 - i_1, i_3 - i_2, \dots, i_k - i_{k-1}, n - i_k}.$$

\square

4. INTERPOLATION FORMULAS

In this section we present a formula that relates the multi-indexed Whitney numbers of the first and second kind. These formulas are very elementary and probably were known before but we could not find them in the literature. Let \mathcal{P} be a locally finite ranked poset with smallest element $\hat{0}$. All the elements of \mathcal{P} of rank k we denote by $\mathcal{P}_k := \{X \in \mathcal{P} | \text{rk} X = k\}$ and for $I = \{i_1, \dots, i_s\}$ we set $\mathcal{P}(I) = \{\vec{X} = (X_1, \dots, X_s) | \forall 1 \leq i \leq s, X_i \in \mathcal{P}(i)\}$. Also, we call $\mathcal{P}_X = \{Y \in \mathcal{P} | Y \leq X\}$ the localization of \mathcal{P} at X and $\mathcal{P}^X = \{Y \in \mathcal{P} | Y \geq X\}$ the restriction of \mathcal{P} at X . Using these new posets we record a few basic lemmas which are foundational for computing various Whitney numbers.

Lemma 4.1. *If $I \subseteq \{1, \dots, n-1\}$ then*

$$\sum_{X \in \mathcal{P}_n} W_I(\mathcal{P}_X) = W_{I \cup \{n\}}(\mathcal{P}).$$

Proof. By definition

$$W_I(\mathcal{P}_X) = \sum_{\substack{\vec{X} \in \mathcal{P}(I) \\ \vec{X} \leq X}} \zeta(\vec{X}).$$

Thus

$$(5) \quad \sum_{X \in \mathcal{P}_n} W_I(\mathcal{P}_X) = \sum_{X \in \mathcal{P}_n} \sum_{\substack{\vec{X} \in \mathcal{P}(I) \\ \vec{X} \leq X}} \zeta(\vec{X}) = \sum_{X \in \mathcal{P}_n} \sum_{\vec{X} \in \mathcal{P}(I)} \zeta(\vec{X}, X).$$

Since the right hand side of 5 is exactly $W_{I \cup \{n\}}$ we are done. \square

We add the “dual” of Lemma 4.1 for later use whose proof is very similar.

Lemma 4.2. *Let $r = \text{rk}(\mathcal{P})$ and $I \subseteq \{1, \dots, t-1\}$. For $I = \{i_1, \dots, i_k\}$ set $I[t] = \{i_1 + t, i_2 + t, \dots, i_k + t\}$ and assume that $i + t \leq r$ for all $i \in I$. With this notation we have*

$$\sum_{X \in \mathcal{P}_t} W_I(\mathcal{P}^X) = W_{\{t\} \cup I[t]}(\mathcal{P}).$$

Proof. In this case the indices I must be shifted to be accounted for in \mathcal{P} because \mathcal{P}^X is all elements above X . So,

$$\begin{aligned} \sum_{X \in \mathcal{P}_t} W_I(\mathcal{P}^X) &= \sum_{X \in \mathcal{P}_t} \sum_{\vec{X} \in \mathcal{P}^X(I)} \zeta(\vec{X}) \\ &= \sum_{X \in \mathcal{P}_t} \sum_{\substack{\vec{X} \in \mathcal{P}(I+t) \\ X \leq \vec{X}}} \zeta(\vec{X}) = \sum_{X \in \mathcal{P}_t} \sum_{\vec{X} \in \mathcal{P}(I+t)} \zeta(X, \vec{X}) = \sum_{\vec{Y} \in (\{t\} \cup (I+t))} \zeta(\vec{Y}) \end{aligned}$$

which is exactly $W_{\{t\} \cup (I+t)}$. \square

We add another lemma for use on understanding the Kazhdan-Lusztig polynomial, which is really a combination of Lemma 4.1 and Lemma 4.2.

Lemma 4.3. *Let $r = \text{rk}(\mathcal{P})$, $k \in [r]$ and $I, J \subseteq [r]$ such that for all $i \in I$, $i \leq k$ and for all $j \in J$, $j + k \leq r$. In this case we can define $J[k] = \{j + k | j \in J\}$. Then*

$$\sum_{F \in L_k} W_I(\mathcal{P}_F) W_J(\mathcal{P}^F) = W_{I \cup \{k\} \cup J[k]}(\mathcal{P}).$$

Proof. Let $I = \{i_1, \dots, i_s\}$ and $J = \{j_1, \dots, j_t\}$. Look at the sum

$$\sum_{F \in L_k} W_I(\mathcal{P}_F) W_J(\mathcal{P}^F) = \sum_{F \in \mathcal{P}_k} \left(\sum_{\mathcal{X}} 1 \right) \left(\sum_{\mathcal{Y}} 1 \right)$$

where the summation condition \mathcal{X} correspond to $X_u \in \mathcal{P}_{i_u}$ for $1 \leq u \leq s$ and $X_{i_1} \leq \dots \leq X_{i_s} \leq F$ and the summation condition \mathcal{Y} corresponds to $Y_v \in \mathcal{P}_{j_v}$ for $1 \leq v \leq t$ and $F \leq Y_{j_1} \leq \dots \leq Y_{j_t}$. Then we switch move the sums together and we have the result. \square

Now we present the main theorem of this section.

Theorem 4.4. *If \mathcal{P} is a locally finite, ranked poset and $1 \leq n \leq \text{rk} \mathcal{P}$ then*

$$w_{0,n} = \sum_{I \subseteq \{1, \dots, n-1\}} (-1)^{|I|+1} W_{I \cup \{n\}}.$$

Proof. We argue by induction. Note that the sum includes the empty set $I = \emptyset$. The base case is $w_{0,1}$. This is clearly $-W_1$. Now

$$\begin{aligned}
 w_{0,n} &= \sum_{X \in \mathcal{P}_n} \mu(\hat{0}, X) = \sum_{X \in \mathcal{P}_n} \left(- \sum_{Y < X} \mu(\hat{0}, Y) \right) \\
 &= \sum_{X \in \mathcal{P}_n} \left[-1 - \sum_{\substack{X_1 \in \mathcal{P}_1 \\ X_1 \leq X}} \mu(\hat{0}, X_1) - \cdots - \sum_{\substack{X_{n-1} \in \mathcal{P}_{n-1} \\ X_{n-1} \leq X}} \mu(\hat{0}, X_{n-1}) \right] \\
 (6) \qquad &= \sum_{X \in \mathcal{P}_n} \left[- \sum_{i=0}^{n-1} w_{0,i}(\mathcal{P}_X) \right]
 \end{aligned}$$

Now we can apply our induction hypothesis to each term of 6:

$$= \sum_{X \in \mathcal{P}_n} \left[- \sum_{i=0}^{n-1} \left[\sum_{I \subset \{1, \dots, i-1\}} (-1)^{|I|+1} W_{I \cup \{i\}}(\mathcal{P}_X) \right] \right]$$

and now switch sums to get

$$\begin{aligned}
 (7) \qquad & - \sum_{i=0}^{n-1} \left[\sum_{I \subset \{1, \dots, i-1\}} (-1)^{|I|+1} \left[\sum_{X \in \mathcal{P}_n} W_{I \cup \{i\}}(\mathcal{P}_X) \right] \right]
 \end{aligned}$$

Then by Lemma 4.1 we have that 7 becomes

$$- \sum_{i=0}^{n-1} \left[\sum_{I \subset \{1, \dots, i-1\}} (-1)^{|I|+1} W_{I \cup \{i\} \cup \{n\}} \right]$$

which finishes the proof. \square

As our first application to Theorem 4.4 we get formulas for the number of regions in the complement of a real hyperplane arrangement.

Corollary 4.5. *Let \mathcal{A} be a real, essential hyperplane arrangement in dimension n and $a(\mathcal{A})$ = number of regions in the complement and $b(\mathcal{A})$ = number of relatively bounded regions in the complement. Then*

$$a(\mathcal{A}) = (-1)^n \sum_{I \subseteq [n]} (-1)^{|I|+u(I)} W_I$$

and

$$b(\mathcal{A}) = (-1)^n \sum_{I \subseteq [n]} (-1)^{|I|} W_I$$

where $u(I)$ is the largest element in I .

Proof. Using Zaslavsky's formula, [24], for the $a(\mathcal{A})$ and $b(\mathcal{A})$ as evaluations of the characteristic polynomial we get

$$a(\mathcal{A}) = (-1)^n \left(\sum_{i=0}^n w_{0,i} \right)$$

and

$$a(\mathcal{A}) = (-1)^n \left(\sum_{i=0}^n (-1)^i w_{0,i} \right).$$

Then substitute the interpolation formula, Theorem 4.4 into each Whitney number of the first kind in these sums and we have the result. \square

5. THE KAZHDAN-LUSZTIG POLYNOMIAL OF A MATROID

We use Theorem 4.4 to get closed formulas for certain coefficients of the Kazhdan-Lusztig polynomial of a matroid. This result gives some hint that these polynomials may be more tractable to understand than the classical Kazhdan-Lusztig polynomials. These matroid Kazhdan-Lusztig polynomials were originally defined for matroids or their lattice of flats. However they can be defined for any finite ranked poset. To do this we need a little notation. Let \mathcal{P} be a finite ranked poset. For $F \in \mathcal{P}$ the restriction of \mathcal{P} to F is

$$\mathcal{P}^F = \{E \in \mathcal{P} \mid E \geq F\}$$

and the localization of \mathcal{P} at F is

$$\mathcal{P}_F = \{E \in \mathcal{P} \mid E \leq F\}.$$

Definition 5.1 ([7] Theorem 2.2). Let \mathcal{P} be a finite ranked poset. The *Kazhdan-Lusztig polynomial* of \mathcal{P} , $P(\mathcal{P}, t)$ is the polynomial recursively defined which satisfies

- (1) If $\text{rk}(\mathcal{P}) = 0$ then $P(\mathcal{P}, t) = 0$.
- (2) If $\text{rk}(\mathcal{P}) > 0$ then $\deg(P(\mathcal{P}, t)) < .5\text{rk}(\mathcal{P})$.
- (3) For all \mathcal{P} ,

$$t^{\text{rk}(\mathcal{P})} P(\mathcal{P}, t^{-1}) = \sum_{F \in \mathcal{P}} \chi_1(\mathcal{P}_F, t) P(\mathcal{P}^F, t)$$

where $\chi_1(\mathcal{P}, t)$ is the usual characteristic polynomial (see [16] or definition 6.1).

Now we gather some basic results on the first few coefficients from [7].

Proposition 5.2 ([7] Propositions 2.11, 2.12, and 2.16). *Let L be a geometric lattice with rank r . Then*

- (1) *The constant coefficient of $P(L, t)$ is 1.*
- (2) *The linear term of $P(L, t)$ is $W_{r-1} - W_1$.*
- (3) *The quadratic term of $P(L, t)$ is*

$$W_{1,2} - W_{1,r-1} + W_{r-3,r-1} - W_{r-3,r-2} + W_{r-2} - W_2.$$

Next we develop some notation to state a formula for any coefficient of the Kazhdan-Lusztig polynomial of any poset \mathcal{P} . We are going to compute the degree k term. Developing the index set to sum over is the hard part. Throughout we denote $\{1, \dots, n\}$ by $[n]$. We are going to define the index set, which we will call S_k , recursively. The base is $S_1 = \{1\}$. For $1 \leq t$ put

$$A_t = \left\{ I \in 2^{[t]} \mid I \cap \{t\} \neq \emptyset \right\}$$

and for $t \leq 0$ we set $A_t = \emptyset$. Next for $3 \leq s \leq 2k - 1$ we need a function $f_s : \mathbb{Z}[r] \rightarrow \mathbb{Z}[r]$ defined by

$$f_s(p) = \text{eval}(p, s) + r - s$$

where $\text{eval}(p, s)$ means evaluating the polynomial p at s . Then for a finite subset I of $\mathbb{Z}[r]$ put $F_s(I) = \{f_s(i) \mid i \in I\}$. Then for $k > 1$ set

$$(8) \quad S_k = A_k \cup \bigcup_{s=3}^{2k-1} T_k^s$$

where

$$(9) \quad T_k^s = \bigcup_{u \leq i < s/2} T_k^s(i)$$

with $u = \max\{1, s - k\}$ and

$$(10) \quad T_k^s(i) = \left\{ \alpha \sqcup \{r - s\} \sqcup F_s(\beta) \mid \alpha \in A_{k-s+i}, \beta \in S_i \right\}.$$

Now we need a technical lemma to finish the remainder of the construction. This lemma is the crux of the entire formula. To prove this lemma we will need a little notation. For $I \in T_k^s$ (or $I \in A_k$ or $F_s(I)$) which has some elements deleted, but not all r 's) let

$$\max^r(I) = \max\{n \in \mathbb{Z} \mid r - n \in I\}$$

and similarly

$$\min^r(I) = \min\{n \in \mathbb{Z} \mid r - n \in I\}.$$

Lemma 5.3. *The recursive construction of S_k makes sense and is injective, meaning that for each $I = \alpha \sqcup \{r - s\} \sqcup F_s(\beta) \in T_k^s$ there does not exist a different α' and β' such that $I = \alpha' \sqcup \{r - s\} \sqcup F_s(\beta')$.*

Proof. First we show that the recursions in the formula make sense. The main index sets S_k are defined for $k \geq 1$. For $k = 1$ $S_1 = \{1\}$ and then we define them recursively thereafter. The recursion in (8), (9), and (10) makes sense for $k > 1$ because when i ranges from $\max(1, s - k)$ to $\lceil s/2 \rceil - 1$ the largest it can be is when $s = 2k - 1$. In this case $\lceil s/2 \rceil - 1 = k - 1$ and all the sets defined in (10) are defined by induction. Also, note that when we define the formula of this theorem we are treating r as a variable in a polynomial ring. So, since every set in T_k^s all contain $r - s$ and A_k does not contain any r we know that differentiating between α and $F_s(\beta)$ in the recursion is well defined.

By induction on $k \geq 2$ we show that

$$(11) \quad \max\{\max^r(I) \mid I \in S_k\} = 2k - 1.$$

The base case when $k = 2$ is done by Proposition 5.2. Now suppose that $k > 2$. Since $\max^r(I) = 0$ for $I \in 2^{[k-1]}$ we may assume that $I \in T_k^s$ for some $3 \leq s \leq 2k - 1$. So, $I = \alpha \sqcup \{r - s\} \sqcup F_s(\beta)$ where $\alpha \in 2^{k-s+i}$ and $\beta \in S_i$ for some $\max\{1, k - s\} \leq i \leq s/2$. Since α has no r variables we only need to consider $F_s(\beta)$. By induction since $i < k$ we have $\max^r(\beta) \leq 2i - 1 < s$. Hence $\max^r(I) = s$ and since the maximum that s can be is $2k - 1$ we have finished proving (11). Notice that within this proof we have also concluded that

$$(12) \quad \max^r(I) = s$$

for $I \in T_k^s$. If $I \in T_k^s$ in the form $I = \alpha \sqcup \{r - s\} \sqcup F_s(\beta)$ since the elements of α are all integers and $F_{s'}(I) = F_{s'}(\alpha) \sqcup \{r - s\} \sqcup F_s(\beta)$ then we can also conclude that

$$(13) \quad \max^r(F_{s'}(I)) = \max\{s, s' - \min(\alpha)\}.$$

Next we show that the recursion in defining S_k is injective by induction. By this we mean that for each $I = \alpha \sqcup \{r - s\} \sqcup F_s(\beta) \in T_k^s$ there does not exist another α' and β' such that $I = \alpha' \sqcup \{r - s\} \sqcup F_s(\beta')$. The base case is when $k = 2$ and this can easily be seen from the formulas in Proposition 5.2. Now suppose $k > 2$ and there was such an α' and β' . Since the α and α' sets only have integer elements (i.e. no r variables) and all other elements contain the variable r then definitely $\alpha = \alpha'$. Also, $\alpha = \alpha' = J \cup \{k - s + i\}$ for some $J \in 2^{k-s+i-1}$. This implies that $|\beta| = |\beta'|$ and $\beta, \beta' \in S_i$ for some $\max\{1, s - k\} \leq i < s/2$. Pairing with (12) we have that $F_s(\beta) = F_s(\beta')$ with $\beta, \beta' \in S_i$. If $\beta, \beta' \in A_i$ then clearly $\beta = \beta'$ since f_s is injective when restricted to just integers.

Suppose $\beta \in A_i$ and $\beta' \in T_i^{s'}$ where $3 \leq s' \leq 2i - 1$. Then $\min^r(F_s(\beta)) = s - i$. Also, there exists $\bar{\alpha} \in A_{i-s'+i'}$ and $\bar{\beta} \in S_{i'}$ such that $\beta' = \bar{\alpha} \sqcup \{r - s'\} \sqcup F_{s'}(\bar{\beta})$ where $\max\{1, s' - i\} \leq i' \leq s'/2$. The function F_s is the identity on elements that are outputs from another function $F_{s'}$. Hence $F_s(\beta') = F_s(\bar{\alpha}) \sqcup \{r - s'\} \sqcup F_{s'}(\bar{\beta})$. Then $\min^r(F_s(\bar{\alpha})) = s - (i - s' + i') > s/2 + s'/2 > i + i' > i' \geq \min^r(F_{s'}(\bar{\beta}))$ by induction. Hence by induction, with the base $i = 1$ clear from Proposition 5.2, on i we have that $\min^r(F_s(\beta')) \leq i$. Since $s - i > i$ we have concluded that it is impossible in this case to have $F_s(\beta) = F_{s'}(\beta')$.

In order to treat this next case we need another general inequality. Suppose $\beta \in T_i^{s'}$ with $\beta = \alpha \sqcup \{r - s'\} \sqcup F_{s'}(\lambda)$ where $\alpha \in A_{i-s'-i'}$, $\lambda \in S_{i'}$, and $\max\{1, s' - i\} \leq i' < s'/2$. Then picking s such that $\max\{1, s - k\} \leq i < s/2$ we will compute $F_s(\beta)$. Note that $\min^r(F_s(\alpha)) = s - (i - s' + i') = s - i + s_1 - i_1 > s/2 + s'/2 > s'$ and $\max^r\{F_s(\beta \setminus \alpha)\} = s'$. Hence

$$(14) \quad \min^r(F_s(\alpha)) > \max^r(F_s(\beta \setminus \alpha)).$$

Combining this with (13) we get

$$(15) \quad \max^r(F_s(\beta)) = s - \min(\alpha).$$

Now we can deal with the next case directly. Suppose that $\beta \in T_i^{s_1}$ and $\beta' \in T_i^{s_2}$ where $F_s(\beta) = F_s(\beta')$. So, there exists i_1 and i_2 satisfying $\max\{1, s_1 - i\} \leq i_1 \leq s_1/2$ and $\max\{1, s_2 - i\} \leq i_2 \leq s_2/2$ with $\beta = \alpha_1 \sqcup \{r - s_1\} \sqcup F_{s_1}(\lambda_1)$ and $\beta' = \alpha_2 \sqcup \{r - s_2\} \sqcup F_{s_2}(\lambda_2)$ where $\alpha_1 \in A_{i-s_1+i_1}$, $\alpha_2 \in A_{i-s_2+i_2}$, $\lambda_1 \in S_{i_1}$, and $\lambda_2 \in S_{i_2}$. Now assume that $s_1 \leq s_2$. So, by (14) we have that $\alpha_1 \supseteq \alpha_2$. Then we can consider $\beta \setminus \alpha_2$ and $\beta' \setminus \alpha_2$. If $\alpha_2 \neq \emptyset$

then by a second induction on $|\beta| = |\beta'|$ (with the base case being $|\beta| = 1$ is trivial) we are done. If $\alpha_2 = \emptyset$ then $F_s(\beta') = \beta' = F_s(\alpha_1) \sqcup \{r - s_1\} \sqcup F_{s_1}(\lambda_1)$. If $|\alpha_1| \geq 1$ then $\beta' \setminus \{r - s_2\} = F_s(F_{s_2}(\lambda_2)) = F_s(\alpha_1 \setminus \{\min\{\alpha_1\}\}) \sqcup \{r - s_1\} \sqcup F_{s_1}(\lambda_1)$ and again by induction we are done. If $|\alpha_1| = \emptyset$ then $s_1 = s_2$ and $F_{s_1}(\lambda_2) = F_{s_1}(\lambda_1)$ and again by induction we are done. \square

The set S_k will be the index set which we will sum over. But we need to create a “top heavy” partner for each index set I to get the full formula. To do this we need a function $d : S_k \rightarrow \mathbb{Z}[r]$ defined as follows. For $I \in A_k$ define

$$d(I) = \begin{cases} k & \text{if } I = \{k\} \\ k - \max\{I \setminus \{k\}\} & \text{if } I \neq \{k\} \end{cases}$$

and for $I \in T_k^s$ define

$$d(I) = \min^r(I) - \min^r(I \setminus \{r - \min^r(I)\}).$$

Finally the “top heavy” partner for $I \in S_k$ is

$$t(I) = \begin{cases} I \setminus \{d(I)\} \cup \{r - d(I)\} & \text{if } I \in A_k \\ I \setminus \{r - \min^r(I)\} \cup \{r - d(I)\} & \text{otherwise.} \end{cases}$$

The last piece of the formula we need is a sign function $s_k : S_k \rightarrow \mathbb{Z}$. We also do this recursively. The base is $k = 1$ and we set $s_1(\{1\}) = 0$. For $k > 1$ again we split this up differently for $I \in A_k$ and $I \in \bigcup_{s=3}^{2k-1} T_k^s$. For $I \in A_k$ set $s_k(I) = |I| - 1$. For $I \in T_k^s(i)$ there exists $\alpha \in A_{k-s+i}$ and $\beta \in S_i$ such that $I = \alpha \sqcup \{r - s\} \sqcup F_s(\beta)$ where $s_i(\beta)$ is already defined in the context of S_i . Then set $s_k(I) = |\alpha| + s_i(\beta)$. This makes sense because of Lemma 5.3. Now we state the main theorem of this section.

Theorem 5.4. *For any finite, ranked lattice \mathcal{P} with rank r the degree k coefficient of the Kazhdan-Lusztig polynomial of \mathcal{P} for $1 \leq k < r/2$ is*

$$\sum_{I \in S_k} (-1)^{s_k(I)} (W_{t(I)}(\mathcal{P}) - W_I(\mathcal{P})).$$

Proof. Induct of k . The base $k = 1$ is done in Lemma 5.2. Now we compute the degree k term where $k > 1$. In (3) of the recursion in Definition 5.1 the left hand side has the degree k coefficient on the t^{r-k} term. This is the terms that we will examine on the right hand side. First we split the right hand side up in terms of rank so that we rewrite it as

$$(16) \quad \sum_{s=0}^r \sum_{F \in \mathcal{P}_{r-s}} \chi_1(\mathcal{P}_F, t) P(\mathcal{P}^F, t).$$

Now we reduce this further. Suppose that $s > 2k - 1$. Then for $F \in \mathcal{P}_{r-s}$ $\deg(\chi_1(\mathcal{P}_F, t)) = r - s$ and $\deg(P(\mathcal{P}^F, t)) < s/2$. So, $\deg(\chi_1(\mathcal{P}_F, t)) + \deg(P(\mathcal{P}^F, t)) < r - s + s/2 = r - s/2 \leq$

$r - k$. Hence we can reduce (16) to

$$(17) \quad \sum_{s=0}^{2k-1} \sum_{F \in \mathcal{P}_{r-s}} \chi_1(\mathcal{P}_F, t) P(\mathcal{P}^F, t).$$

Note that since $k < r/2$ and $\deg(P(\mathcal{P}^F, t)) = s$ we know that the coefficients $P(\mathcal{P}^F, t)$ will all be computed by induction. For any polynomial p let $u(i, p)$ denote the coefficient of the i^{th} term and $d(i, p)$ be the i^{th} term down from the top term (i.e. if $\deg p = d$ then $d(i, p) = u(d - i, p)$). Then for each term in (17) the possible products which will yield a degree $r - k$ term are of the form

$$d(k - s + i, \chi_1(\mathcal{P}_F, t)) u(i, P(\mathcal{P}^F, t))$$

where $\max\{0, s - k\} \leq i < s/2$. Hence the total coefficient we are seeking is

$$(18) \quad \sum_{s=0}^{2k-1} \sum_{F \in \mathcal{P}_{r-s}} \sum_{\substack{\max\{0, s-k\} \\ \leq i < s/2}} d(k - s + i, \chi_1(\mathcal{P}_F, t)) u(i, P(\mathcal{P}^F, t)).$$

We first focus on the terms where $i = 0$. For these terms we have by Proposition 5.2

$$u(0, P(\mathcal{P}^F, t)) = 1$$

and

$$(19) \quad d(k - s, \chi_1(\mathcal{P}_F, t)) = w_{0, k-s}(\mathcal{P}_F).$$

Then we use Theorem 4.4 on (19) to get

$$(20) \quad d(k - s, \chi_1(\mathcal{P}_F, t)) = \sum_{I \subseteq \{1, \dots, k-s-1\}} (-1)^{|I|+1} W_{I \cup \{k-s\}}(\mathcal{P}_F).$$

Notice that $0 \leq s \leq k$. Summing over $F \in \mathcal{P}_{r-s}$ and applying Lemma 4.1 to (20) we get

$$(21) \quad \sum_{F \in \mathcal{P}_{r-s}} d(k - s, \chi_1(\mathcal{P}_F, t)) = \sum_{I \subseteq \{1, \dots, k-s-1\}} (-1)^{|I|+1} W_{I \cup \{k-s\} \cup \{r-s\}}(\mathcal{P})$$

as long as $s \neq 0$. In that case since \mathcal{P} is a lattice the sum only contains one term where $\mathcal{P}_F = \mathcal{P}$. Hence

$$(22) \quad d(k, \chi_1(\mathcal{P}, t)) = \sum_{I \subseteq \{1, \dots, k-1\}} (-1)^{|I|+1} W_{I \cup \{k\}}(\mathcal{P}).$$

The subscripts of (22) give exactly all the terms of A_k as well as the signs $s_k(I)$ for $I \in A_k$. Finally for each $I \subseteq A_k$ with $I = J \cup \{k - s\} \cup \{k\}$ for $1 \leq s < k$ there is exactly one term in (21), that being $J \cup \{k - s\} \cup \{r - s\}$ which is the top heavy pair to I . Also note that this exactly covers all the terms of (21) and (22). Hence we have verified the formula for $i = 0$ and equivalently A_k .

Next we focus on the case where $1 \leq i$. Since $i < s/2 \leq k$ we have by induction that

$$(23) \quad u(i, P(\mathcal{P}^F, t)) = \sum_{I \in S_i} (-1)^{s_i(I)} (W_{t(I)}(\mathcal{P}^F) - W_I(\mathcal{P}^F)).$$

Because $F \in \mathcal{P}_{r-s}$ we know $\text{rk}(\mathcal{P}^F) = s$ and so when using this induction all the subscripts in the formula have r replaced with s . The terms coming from the characteristic polynomial are

$$(24) \quad d(k-s+i, \chi_1(\mathcal{P}_F, t)) = w_{0, k-s+i}(\mathcal{P}_F).$$

Again using Theorem 4.4 (24) becomes

$$(25) \quad d(k-s+i, \chi_1(\mathcal{P}_F, t)) = \sum_{\alpha \in A_{k-s+i}} (-1)^{|\alpha|} W_\alpha(\mathcal{P}_F).$$

Next putting (23) and (25) together for each $i > 0$ term of (18) we get

$$(26) \quad \sum_{s=0}^{2k-1} \left[\sum_{\alpha \in A_{k-s+i}} (-1)^{|\alpha|} W_\alpha(\mathcal{P}_F) \right] \left[\sum_{\beta \in S_i} (-1)^{s_i(\beta)} (W_{t(\beta)}(\mathcal{P}^F) - W_\beta(\mathcal{P}^F)) \right].$$

Moving sums together (26) becomes

$$(27) \quad \sum_{s=0}^{2k-1} \sum_{\alpha \in A_{k-s+i}} \sum_{\beta \in S_i} (-1)^{|\alpha|+s_i(\beta)} (W_\alpha(\mathcal{P}_F) W_{t(\beta)}(\mathcal{P}^F) - W_\alpha(\mathcal{P}_F) W_\beta(\mathcal{P}^F)).$$

Then summing over $F \in \mathcal{P}_{r-s}$ and applying Lemma 4.3 to (27) we get

$$(28) \quad \sum_{s=0}^{2k-1} \sum_{\alpha \in A_{k-s+i}} \sum_{\beta \in S_i} (-1)^{|\alpha|+s_i(\beta)} (W_{\alpha \cup \{r-s\} \cup t(\beta)[r-s]}(\mathcal{P}) - W_{\alpha \cup \{r-s\} \cup \beta[r-s]}(\mathcal{P})).$$

This all makes sense because inside \mathcal{P}^F the rank is s and all the elements of every β above are $< s$. Hence in the total lattice \mathcal{P} we can add $r-s$ and satisfy the hypothesis of Lemma 4.3. Finally to finish the proof note that $t(\beta)[r-s] = t(F_s(\beta))$ and $\beta[r-s] = F_s(\beta)$. \square

In Table 2 we print the formula from Theorem 5.4 for $k = \{1, 2, 3, 4, 5\}$. For $k = 6$ the formula takes up too much space. This was calculated using Sage (see [4]). There we only list the index set $S(k)$ and the corresponding sign $s_k(I)$.

Remark 5.5. Note that Lemma 5.3 shows that there is exactly one term of each type in the formula of Theorem 5.4.

Remark 5.6. Notice that if the poset is a geometric lattice each term in Theorem 5.4 is conjectured to be positive and is called the "top heaviness conjecture" (see [14]). Also, it is conjectured that each of the coefficients themselves are conjectured to be positive for matroids (equivalently geometric lattices) (see [7]). However many of the signs s_k are negative and at the moment we do not see a general relationship between these conjectures other than the formula of Theorem 5.4.

Remark 5.7. Theorem 5.4 provides an automatic proof to one implication in Proposition 2.14 of [7] which says that a modular lattice L has $P(L, t) = 1$.

k	$s_k(I)I$ for I in $S(k)$
1	$+ [1]$
2	$+ [2], + [r-3, r-2], - [1, 2]$
3	$+ [3], - [1, 3], - [2, 3], + [1, 2, 3], - [1, r-3, r-2], + [r-4, r-3], + [r-5, r-3],$ $+ [r-5, r-3, r-2], - [r-5, r-4, r-3]$
4	$+ [4], - [1, 4], - [2, 4], + [1, 2, 4], - [3, 4], + [1, 3, 4], + [2, 3, 4], - [1, 2, 3, 4],$ $- [2, r-3, r-2], - [1, r-4, r-3], + [r-5, r-4], + [r-6, r-4],$ $+ [r-6, r-3, r-2], - [r-6, r-5, r-4], + [r-7, r-4], - [r-7, r-6, r-4],$ $- [r-7, r-5, r-4], + [r-7, r-6, r-5, r-4], - [r-7, r-6, r-3, r-2],$ $+ [r-7, r-4, r-3], + [r-7, r-5, r-3], + [r-7, r-5, r-3, r-2],$ $- [r-7, r-5, r-4, r-3]$
5	$+ [5], - [1, 5], - [2, 5], + [1, 2, 5], - [3, 5], + [1, 3, 5], + [2, 3, 5], - [1, 2, 3, 5],$ $- [4, 5], + [1, 4, 5], + [2, 4, 5], - [1, 2, 4, 5], + [3, 4, 5], - [1, 3, 4, 5], - [2, 3, 4, 5],$ $+ [1, 2, 3, 4, 5], - [3, r-3, r-2], - [2, r-4, r-3], - [1, r-5, r-4], + [r-6, r-5],$ $+ [r-7, r-5], + [r-7, r-3, r-2], - [r-7, r-6, r-5], + [r-8, r-5],$ $- [r-8, r-7, r-5], - [r-8, r-6, r-5], + [r-8, r-7, r-6, r-5],$ $- [r-8, r-7, r-3, r-2], + [r-8, r-4, r-3], + [r-8, r-5, r-3],$ $+ [r-8, r-5, r-3, r-2], - [r-8, r-5, r-4, r-3], + [r-9, r-5],$ $- [r-9, r-8, r-5], - [r-9, r-7, r-5], + [r-9, r-8, r-7, r-5],$ $- [r-9, r-6, r-5], + [r-9, r-8, r-6, r-5], + [r-9, r-7, r-6, r-5],$ $- [r-9, r-8, r-7, r-6, r-5], - [r-9, r-7, r-3, r-2], - [r-9, r-8, r-4, r-3],$ $+ [r-9, r-5, r-4], + [r-9, r-6, r-4], + [r-9, r-6, r-3, r-2],$ $- [r-9, r-6, r-5, r-4], + [r-9, r-7, r-4], - [r-9, r-7, r-6, r-4],$ $- [r-9, r-7, r-5, r-4], + [r-9, r-7, r-6, r-5, r-4],$ $- [r-9, r-7, r-6, r-3, r-2], + [r-9, r-7, r-4, r-3],$ $+ [r-9, r-7, r-5, r-3], + [r-9, r-7, r-5, r-3, r-2],$ $- [r-9, r-7, r-5, r-4, r-3]$

TABLE 2. Low degree coefficient formulas for the matroid KL polynomial

6. GENERALIZED CHARACTERISTIC POLYNOMIALS

Using these new Whitney numbers one can define new polynomials.

Definition 6.1. The k^{th} generalized characteristic polynomial of a finite ranked poset \mathcal{P} with smallest element $\hat{0}$ is

$$\chi_k(\mathcal{P}, \mathbf{t}) = \sum_{|I|=k} w_{\{\hat{0}\} \cup I} \mathbf{t}^I$$

where $\mathbf{t}^I = t_1^{i_1} t_2^{i_2} \dots t_k^{i_k}$ when $I = \{i_1, \dots, i_k\}$. For an arrangement \mathcal{A} we define the k^{th} characteristic polynomial

$$\chi_k(\mathcal{A}, \mathbf{t}) = \sum_{(X_1, \dots, X_k) \in \mathcal{F}l^k(L(\mathcal{A}))} \mu_{k+1}(\hat{0}, X_1, \dots, X_k) t_1^{\dim X_1} \dots t_k^{\dim X_k}.$$

Remark 6.2. The usual characteristic polynomial of a poset \mathcal{P} is exactly $\chi_1(\mathcal{P}, t) = \chi(\mathcal{P}, t)$.

We get the following result from viewing χ_k as an element of the incidence algebra $\mathcal{I}^k(\mathcal{P}, \mathbb{Z}[t_1, \dots, t_k])$ and Proposition 2.7.

Proposition 6.3. $\chi_k(P \times Q; t_1, \dots, t_k) = \chi_k(P; t_1, \dots, t_k) \chi_k(Q; t_1, \dots, t_k)$.

Using this we can compute the characteristic polynomial of the Boolean lattices. Let \mathcal{B}_n be the Boolean lattice of rank n . Again we use the product formula on $\mathcal{B}_n \cong (\mathcal{B}_1)^n$. For $2 \leq i \leq k+1$ let $\tilde{w}_i = w_{0, \dots, 0, 1, \dots, 1}(\mathcal{B}_1)$ where there are i 1's. Then by Proposition 3.4 we have that $\tilde{w}_i = (-1)^i$. Thus

$$(29) \quad \chi_k(\mathcal{B}_1; t_1, \dots, t_k) = \sum_{i=0}^k (-1)^i \prod_{j=1}^{k-i} t_j$$

$$(30) \quad = t_1(t_2(\dots(t_{k-1}(t_k - 1) + 1) \dots + (-1)^{k-1}) + (-1)^k).$$

Applying Proposition 6.3 to Equations (29) and (30) we get the following proposition.

Proposition 6.4. *The characteristic polynomials of the Boolean matroid are*

$$\begin{aligned} \chi_k(\mathcal{B}_n; t_1, \dots, t_k) &= \left(\sum_{i=0}^k (-1)^i \prod_{j=1}^{k-i} t_j \right)^n \\ &= \left(t_1(t_2(\dots(t_{k-1}(t_k - 1) + 1) \dots + (-1)^{k-1}) + (-1)^k \right)^n \end{aligned}$$

where the last term in the sum of products the product term is 1.

Using Proposition 3.4, Proposition 3.6, and the Proposition 6.4 we get a formula for multinomial coefficients.

Corollary 6.5.

$$\begin{aligned} &\left(t_1(t_2(\dots(t_{k-1}(t_k - 1) + 1) \dots + (-1)^{k-1}) + (-1)^k \right)^n \\ &= \sum_{\{i_1, \dots, i_k\} \subset [n]} (-1)^{i_1 + \dots + i_k} \binom{n}{i_1, i_2 - i_1, i_3 - i_2, \dots, i_k - i_{k-1}, n - i_k} t_1^{i_1} \dots t_k^{i_k}. \end{aligned}$$

Next, using Proposition 6.4, we note that k^{th} generalized characteristic polynomial for the the Boolean poset satisfies a nice identity relating to the classical characteristic polynomial.

Corollary 6.6.

$$\chi_k(\mathcal{B}_n; t_1, \dots, t_k) = (-1)^n \chi_1(\mathcal{B}_n; -(\chi_k(\mathcal{B}_1; t_1, \dots, t_k) + (-1)^{k-1})).$$

Now we show that these higher characteristic polynomials do not satisfy a deletion-restriction formula for hyperplane arrangements. First we examine the Boolean formula to deduce what a deletion restriction formula would look like. Let \mathcal{B}_n be the Boolean arrangement in a vector space of rank n and \mathcal{B}'_n and \mathcal{B}''_n be the deletion and restriction respectively by one of the hyperplanes. Note that the deletion of the Boolean arrangement, \mathcal{B}'_n , is a Boolean arrangement of rank $n-1$ but it is just embedded in one higher dimension than needed. The restricted Boolean arrangement, \mathcal{B}''_n , is also Boolean of ranks $n-1$. So,

$$(31) \quad \chi_k(\mathcal{B}'_n; t_1, \dots, t_k) = t_1 \cdots t_k \left(t_1(t_2(\cdots (t_{k-1}(t_k - 1) + 1) \cdots + (-1)^{k-1}) + (-1)^k \right)^{n-1}$$

and

$$(32) \quad \chi_k(\mathcal{B}''_n; t_1, \dots, t_k) = \left(t_1(t_2(\cdots (t_{k-1}(t_k - 1) + 1) \cdots + (-1)^{k-1}) + (-1)^k \right)^{n-1}$$

Hence if we were to have a deletion restriction formula then it would have to be of the form

$$(33) \quad \chi_k(\mathcal{A}; t_1, \dots, t_k) = \chi_k(\mathcal{A}'; t_1, \dots, t_k) - \left(\sum_{i=0}^{k-1} (-1)^i \prod_{j=1}^{k-1-i} t_j \right) \chi_k(\mathcal{A}''; t_1, \dots, t_k)$$

where \mathcal{A}' and \mathcal{A}'' are the deletion and restriction respectively. Note that if (33) were true it would generalize the usual deletion-restriction formula with $k=1$ (see [16]).

However, the next example shows that this formula is not satisfied even for $k=2$ on a rank 2 matroid.

Example 6.7. Let \mathcal{A} be the arrangement of 3 hyperplanes in rank 2. So, \mathcal{A} has intersection lattice of that in Example 2.4. Then the 2nd characteristic polynomial is

$$\chi_2(\mathcal{A}; t_1, t_2) = t_1^2 t_2^2 - 3t_1^2 t_2 + 2t_1^2 + 3t_1 t_2 - 6t_1 + 4.$$

Since the deletion \mathcal{A}' is Boolean of rank 2 we have that $\chi_2(\mathcal{A}'; t_1, t_2) = (t_1(t_2 - 1) - 1)^2$. The restriction \mathcal{A}'' is \mathcal{B}_1 hence $\chi_2(\mathcal{A}''; t_1, t_2) = t_1(t_2 - 1) + 1$. Now if we insert these into the formula (33) we get

$$\begin{aligned} \chi_2(\mathcal{A}'; t_1, t_2) - (t_1 - 1)\chi_2(\mathcal{A}''; t_1, t_2) &= (t_1(t_2 - 1) + 1)^2 - (t_1 - 1)(t_1(t_2 - 1) + 1) \\ &= t_1^2 t_2^2 - 3t_1^2 t_2 + 2t_1^2 + 3t_1 t_2 - 4t_1 + 2 \end{aligned}$$

which has the last two terms different than $\chi_2(\mathcal{A}; t_1, t_2)$.

Remark 6.8. It seems interesting that this fails for such a simple example. However, we regard the failing of a deletion-restriction formula as a good sign. Otherwise, since the characteristic polynomial satisfies the product formula Proposition 6.3, it would be an evaluation of the Tutte polynomial (see for example [2]). In this sense these characteristic polynomials are new invariants.

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